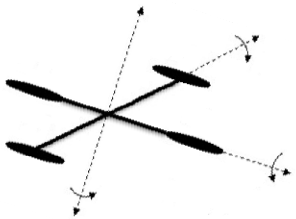
3-1. 선형방정식



■ Initial-Value Problem

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

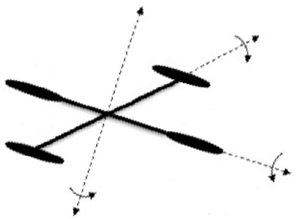
$$I.C.: y(x_0) = y_0, \quad y'(x_0) = y_1, \; \ldots \;, y^{(n-1)}(x_0) = y_{n-1}.$$

Example 1 Unique Solution of an IVP

$$3y''' + 5y'' - y' + 7y = 0, \quad y(1) = 0, \quad y'(1) = 0, \quad y''(1) = 0$$

(Solution)

$$y = 0 \rightarrow x = 1$$



Example 2 Unique Solution of an IVP

$$y'' - 4y = 12x, \quad y(0) = 4, \quad y'(0) = 1$$

(Solution)

- $y = 3e^{2x} + e^{-2x} - 3x$

Boundary-Value Problem

Solve: $a_2(x)\dfrac{d^2y}{dx^2} + a_1(x)\dfrac{dy}{dx} + a_0(x)y = g(x)$

Subject to: $y(a) = y_0, \quad y(b) = y_1$

$$y'(a) = y_0, \quad y(b) = y_1$$

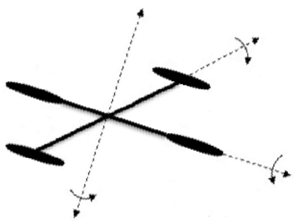
$$y(a) = y_0, \quad y'(b) = y_1$$

$$y'(a) = y_0, \quad y'(b) = y_1,$$

General

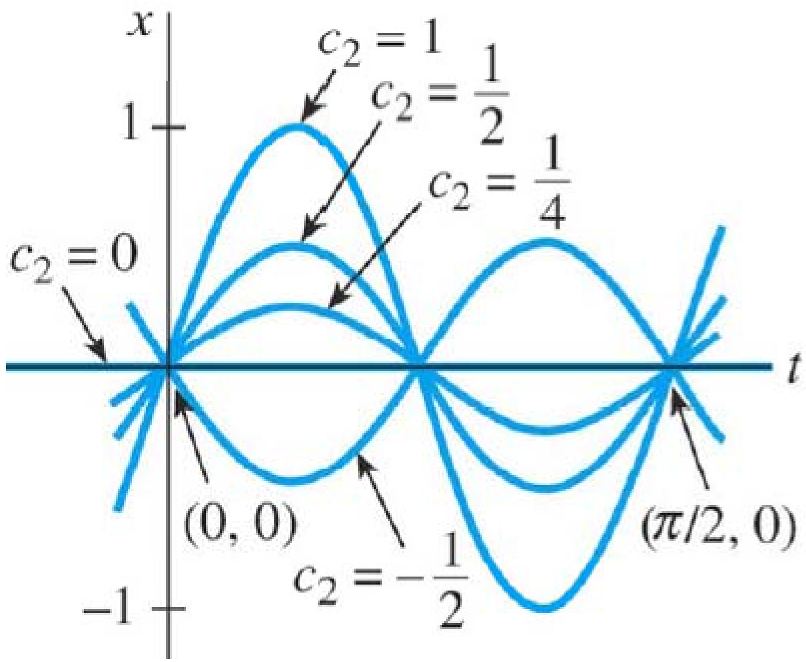
$$\alpha_1 y(a) + \beta_1 y'(a) = \gamma_1$$

$$\alpha_2 y(b) + \beta_2 y'(b) = \gamma_2.$$



Example 3 A BVP Can Have Many, One, or No Solutions

$x'' + 16x = 0 \rightarrow x = c_1 \cos 4t + c_2 \sin 4t.$



(a) $x'' + 16x = 0, \quad x(0) = 0, \quad x(\pi/2) = 0$

B.C. $x(0) = 0 \rightarrow c_2 = 0$

$x = c_1 \cos 4t$

→ infinitely many solutions

(b) $x'' + 16x = 0, \quad x(0) = 0, \quad x(\pi/8) = 0,$

B.C. $x(0) = 0 \rightarrow c_2 = 0 \rightarrow x = c_1 \cos 4t; \; x(\pi/8) = 0 \rightarrow c_1 = 0 \rightarrow y = 0$

→ only solution

(c) $x'' + 16x = 0, \quad x(0) = 0, \quad x(\pi/2) = 1,$

B.C. $x(0) = 0 \rightarrow c_2 = 0 \rightarrow x = c_1 \cos 4t; \; x(\pi/8) = 0 \rightarrow 1 = c_2 \sin 2\pi = c_2 \cdot 0 = 0$ (모순!)

→ no solution

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$ → **Homogeneous**

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$ → **Nonhomogeneous**

Differential Operator: $dy/dx \equiv Dy$

ex: $D(\cos 4x) = -4\sin 4x, \quad D(5x^3 - 6x^2) = 15x^2 - 12x$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D(Dy) = D^2 y \quad \text{and in general} \quad \frac{d^n y}{dx^n} = D^n y,$$

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x).$$

$L\{\alpha f(x) + \beta g(x)\} = \alpha L(f(x)) + \beta L(g(x))$, (Linear Differential Operator)

Differential Equations

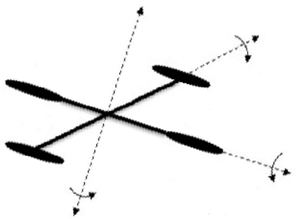
$$y'' + 5y' + 6y = 5x - 3$$

$$\rightarrow \quad D^2 y + 5Dy + 6y = 5x - 3$$

$$\rightarrow \quad (D^2 + 5D + 6)y = 5x - 3$$

$$\rightarrow \quad L(y) = 5x - 3$$

$L(y) = 0$ and $L(y) = g(x)$,



Superposition Principle

Theorem 3.1.2 Superposition Principle – Homogeneous Equations

$$y_1(x),\ y_2(x),\ \cdots,\ y_k(x)$$

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x),$$

Proof

$$\begin{aligned} L(y) &= L\{c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)\} \\ &= c_1 L(y_1) + c_2 L(y_2) + \cdots + c_k L(y_k) \\ &= c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_k \cdot 0 = 0. \end{aligned}$$

Example 4 Superposition – Homogeneous DE

$y_1 = x^2, \;\; y_2 = x^2 \ln x \rightarrow y = c_1 x^2 + c_2 x^2 \ln x \;\; \boxed{x^3 y''' - 2xy' + 4y = 0}$

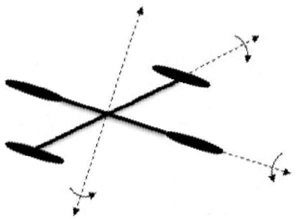
$y = c_1 x^2 + c_2 x^2 \ln x$

$y = e^{7x} \rightarrow \;\; y'' - 9y' + 14y = 0$

$y = 9e^{7x}$

$y = 0$

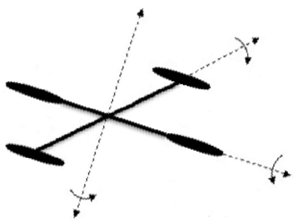
$y = -\sqrt{5} e^{7x}$



$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

y_p → Particular Solution

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x) + y_p$$



Example 10 General Solution of a Nonhomogeneous DE

$$y''' - 6y'' + 11y' - 6y = 3x.$$

$$\rightarrow \ y_p = -(11/12) - (1/2)x$$

$$y''' - 6y'' + 11y' - 6y = 0.$$

$$\rightarrow \ y_c = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$$

(11) general solution

$$y = y_c + y_p = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} - \frac{11}{12} - \frac{1}{2}x.$$

Example 11 Superposition – Nonhomogeneous DE

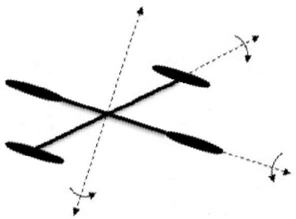
$y_{p_1} = -4x^2$ 은 $y'' - 3y' + 4y = -16x^2 + 24x - 8$

$y_{p_2} = e^{2x}$ 은 $y'' - 3y' + 4y = 2e^{2x}$

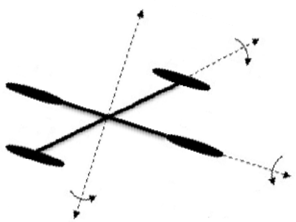
$y_{p_3} = xe^{x}$ 은 $y'' - 3y' + 4y = 2xe^{x} - e^{x}$

$\rightarrow \; y = y_{p_1} + y_{p_2} + y_{p_3} = -4x^2 + e^{2x} + xe^{x},$

$$y'' - 3y' + 4y = \underbrace{-16x^2 + 24x - 8}_{g_1(x)} + \underbrace{2e^{2x}}_{g_2(x)} + \underbrace{2xe^{x} - e^{x}}_{g_3(x)}.$$



3-2. 계수낮추기



Example 1 Finding a Second Solution

$$y'' - y = 0$$

Solution

$$y = u(x)y_1(x) = u(x)e^x$$

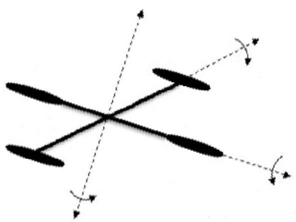
$$y' = ue^x + e^x u', \qquad y'' = ue^x + 2e^x u' + e^x u'',$$

$$y'' - y = e^x(u'' + 2u') = 0. \rightarrow u'' + 2u' = 0.$$

$$w = u' = c_1 e^{-2x} \rightarrow u = -(1/2)c_1 e^{-2x} + c_2$$

$$y = u(x)e^x = -\frac{c_1}{2}e^{-x} + c_2 e^x .; \leftarrow \quad c_1 = -2,\ c_2 = 0$$

$$y_2 = e^{-x}$$



General Case

(1) $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \leftarrow a_2(x)$

$$y'' + P(x)y' + Q(x)y = 0,$$

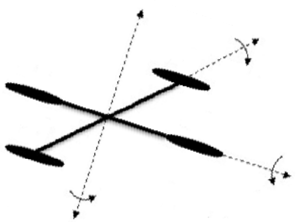
$$y = u(x)y_1(x)$$

$$y' = uy_1' + y_1u', \quad y'' = uy_1'' + 2y_1'u' + y_1u'' \rightarrow (3)$$

$$y'' + Py' + Qy = u[\underbrace{y_1'' + Py_1' + Qy_1}_{\text{zero}}] + y_1u'' + (2y_1' + Py_1)u' = 0.$$

$$y_1u'' + (2y_1' + Py_1)u' = 0 \quad \text{or} \quad y_1w' + (2y_1' + Py_1)w = 0,$$

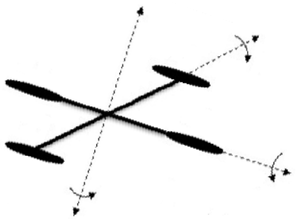
$$\frac{dw}{w} + 2\frac{y_1'}{y_1}dx + P\,dx = 0$$



$$\ln\left|wy_1^2\right| = -\int P\,dx + c \quad \text{or} \quad wy_1^2 = c_1 e^{-\int P\,dx}.$$

$$u = c_1 \int \frac{e^{-\int P\,dx}}{y_1^2}\,dx + c_2.$$

$$y_2 = y_1(x)u(x) = y_1(x)\int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx.$$



$$y_2 = y_1(x)u(x) = y_1(x)\int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx.$$

Example 2 A Second Solution by Formula (5)

Solution

Standard form; $y'' - \dfrac{3}{x}y' + \dfrac{4}{x^2}y = 0,$

$$y_1 = x^2$$

$$y_2 = x^2 \int \frac{e^{3\int dx/x}}{x^4}\,dx = x^2 \int \frac{dx}{x} = x^2 \ln x.$$

General solution

$$y = c_1 x^2 + c_2 x^2 \ln x.$$